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Schrödinger and Dirac Quantum Random Walks

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Quantum random walks, whose amplitude evolutions are given by generalizations of discrete versions of Schrödinger and Dirac equations, are constructed. The results are given in three dimensions and it is shown that they cannot be reduced to stochastically independent one-dimensional motions. Properties of these quantum random walks are analyzed and expressions for their characteristic functions and free propagators are derived.

1. INTRODUCTION

It is well known that Brownian motion is associated with the diffusion and heat equations. Since Brownian motion is a limit of random walks, a random walk is associated with discrete versions of these equations. One might ask whether an analogous situation occurs for the Schrödinger and Dirac equations. Nakamura (1991) has recently shown that there exists a quantum stochastic process whose associated evolution is given by the free one-dimensional Schrödinger equation. Moreover, he proved an analogous result for the one-dimensional Dirac equation.

In the present work, we consider generalizations of discrete versions of the three-dimensional Schrödinger and Dirac equations and construct quantum random walks whose amplitude evolutions are given by these equations. We analyze the properties of these quantum random walks and derive expressions for their characteristic functions and free propagators. For other approaches to those problems we refer the reader to the references in Nakamura (1991). For a general discussion of quantum stochastic processes, the reader may want to consult Gudder and Schindler (1991, 1992) and Marbeau and Gudder (1989, 1990).

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2. SCHRÖDINGER RANDOM WALKS

Let a be a positive real number, N a positive integer, and $\varepsilon = 1/N$. We define the finite discrete timeline T by

$$T = \{0, \varepsilon, 2\varepsilon, \ldots, N^2\varepsilon\} = \left\{0, \frac{1}{N}, \frac{2}{N}, \ldots, N\right\}$$

and the discrete space lattice

$$Q^3 = (a\varepsilon)^{1/2}\mathbb{Z}^3 = \left(\frac{a}{N}\right)^{1/2} \mathbb{Z}^3$$

We now construct a quantum random walk on the spacetime lattice $L = T \times Q^3$. The sample space Ω is given by

$$\Omega = [(\{1, 2, 3\} \times \{-1, 1\}) \cup \{0\}]^T$$

For $t \in T$, $\omega \in \Omega$, if $\omega(t) \neq 0$, we write $\omega(t) = (\omega^1(t), \omega^2(t))$, where $\omega^1(t) \in \{1, 2, 3\}$ and $\omega^2(t) \in \{-1, 1\}$. For $r = 1, 2, 3$, we define $X^r : T \times \Omega \to Q$ by

$$X^r(k\varepsilon, \omega) = (a\varepsilon)^{1/2} \sum_{j=0}^{k-1} \{\omega^2(j\varepsilon) : \omega(j\varepsilon) \neq 0, \omega^1(j\varepsilon) = r\}$$

if $k \neq 0$ and $X^r(0, \omega) = 0$. Finally, we define the random walk $X : T \times \Omega \to Q^3$ by

$$X(k\varepsilon, \omega) = (X^1(k\varepsilon, \omega), X^2(k\varepsilon, \omega), X^3(k\varepsilon, \omega))$$

The random walk X has the following physical interpretation. At each time step $t \in T$, toss a seven-sided die whose faces are labeled by the elements of the set

$$S = (\{1, 2, 3\} \times \{-1, 1\}) \cup \{0\}$$

A particle begins at the origin at time $t = 0$. If the toss at time t results in 0, the particle pauses (does not change position); if the toss results in $(r, \pm 1)$, the particle moves $\pm(a\varepsilon)^{1/2}$ units in the x^r direction, $r = 1, 2, 3$. In the sequel, we shall need the notation $|A|$ for the cardinality of a set A.

Until now we could just as well be describing a classical random walk since we have not yet placed a measure on Ω. In order to describe a quantum random walk, we define an amplitude function on Ω instead of a probability measure. Let $\alpha_j, j = 0, \ldots, 6$, be complex numbers satisfying $\sum_{j=0}^{6} \alpha_j = 1$ and let $f' : S \to \mathbb{C}$ be defined by $f'(0) = \alpha_0$, $f'((j, 1)) = \alpha_j$, $f'((j, -1)) = \alpha_{j+3}$, $j = 1, 2, 3$. Define the *amplitude function* $f : \Omega \to \mathbb{C}$ by

$$f(\omega) = f'[\omega(N^2\varepsilon)] f'[\omega((N^2 - 1)\varepsilon)] \cdots f'[\omega(\varepsilon)] f'[\omega(0)]$$

It follows by induction that for any distinct $r_1\varepsilon, \ldots, r_n\varepsilon \in T$ we have

$$\sum_{\omega} f'[\omega(r_1\varepsilon)] \cdots f'[\omega(r_n\varepsilon)] = 1 \tag{2.1}$$

For a set $A \subseteq \Omega$, we define the *amplitude* $f(A)$ of A by $f(A) = \sum_{\omega \in A} f(\omega)$. Applying (2.1) gives $f(\Omega) = 1$. Moreover, it is clear that $f: 2^\Omega \to \mathbb{C}$ is an additive complex measure.

The map $X: T \times \Omega \to Q^3$ describes a quantum particle with initial position 0. For $y \in Q^3$, the map $y + X: T \times \Omega \to Q^3$ defined by

$$(y+X)(t, \omega) = y + X(t, \omega)$$

describes a quantum particle with initial position y. For $t \in T$, the map $y + X_t: \Omega \to Q^3$ defined by $(y + X_t)(\omega) = y + X(t, \omega)$ is called the *position measurement at time t*. Notice that $y + X_t$ corresponds to a classical random variable. For $\omega \in \Omega$, the map $y + X_\omega: T \to Q^3$ defined by $(y + X_\omega)(t) = y + X(t, \omega)$ is called a *path starting at y*.

Let $V: \mathbb{R}^3 \to \mathbb{R}$ be a function which we view as a potential energy. For $t = k\varepsilon \in T$, we define the *propagator* $K_{V,t}: Q^3 \times \Omega \to \mathbb{C}$ by

$$K_{V,t}(y, \omega) = f(\omega) \prod_{j=0}^{k-1} [1 - i\varepsilon V(y + X_\omega(j\varepsilon))]$$

for $k \neq 0$ and $K_{V,0}(y, \omega) = f(\omega)$. We interpret $K_{V,t}(y, \omega)$ as the amplitude that a particle under the influence of the potential V moves along the path $y + X_\omega$ during the time interval from 0 to t. Notice that for $V = 0$, the *free propagator* $K_{0,t}$ becomes $K_{0,t}(y, \omega) = f(\omega)$. Let $g: \mathbb{R}^3 \to \mathbb{C}$ be a function which we view as the initial amplitude. We then define the *evolution* $U_{V,t}g: Q^3 \to \mathbb{C}$ by

$$(U_{V,t}g)(x) = \sum_{y,\omega} \{K_{V,t}(y, \omega) g(y) : y \in Q^3, y + X_\omega(t) = x\} \tag{2.2}$$

Notice that there are only a finite number of terms in the summation (2.2), since there are only finitely many starting points $y \in Q^3$ that can reach x in time t along a path. We interpret $(U_{V,t}g)(x)$ as the amplitude that a particle is at x at time t when the initial amplitude is g. The function $U_{V,t}g$ gives the *amplitude distribution* of the measurement X_t. Let $\delta_0: \mathbb{R}^3 \to \mathbb{C}$ be defined by

$$\delta_0(y) = \begin{cases} 1 & \text{if } y = 0 \\ 0 & \text{otherwise} \end{cases}$$

We use the notation $f_{V,t}(x) = (U_{V,t}\delta_0)(x)$. It follows from (2.2) that

$$f_{0,t}(x) = \sum_{\omega} \{f(\omega) : X_\omega(t) = x\}$$

As is usual in quantum probability, we interpret $R(t, x) = |(U_{V,t}g)(x)|^2$ as the relative probability that a particle is at position x at time t. Without a further assumption on g, we cannot normalize $R(x, t)$ to obtain an absolute probability. However, if we assume that g has compact support, then $R(t, x)$ vanishes except for finitely many $x \in Q^3$. We can then define the probability that a particle is at x at time t as

$$P(t, x) = \frac{R(t, x)}{\sum_y R(t, y)}$$

We shall soon show that the evolution satisfies a discrete version of Schrödinger's equation. To see that this is reasonable, let us roughly consider what happens when $N \to \infty$ or $\varepsilon \to 0$. For simplicity, suppose $g = \delta_0$. Assuming V is continuous, for ε small we have

$$1 - i\varepsilon V(X_\omega(j\varepsilon)) \approx \exp[-i\varepsilon V(X_\omega(j\varepsilon))]$$

Hence,

$$\begin{aligned} K_{V,t}(0, \omega) &\approx f(\omega) \prod_{j=0}^{k-1} \exp[-i\varepsilon V(X_\omega(j\varepsilon))] \\ &= f(\omega) \exp\left[-i \sum_{j=0}^{k-1} \varepsilon V(X_\omega(j\varepsilon))\right] \\ &\approx f(\omega) \exp\left[-i \int_0^t V(X_\omega(\tau))\, d\tau\right] \end{aligned}$$

We then have

$$f_{V,t}(x) \approx \sum_\omega \left\{ f(\omega) \exp\left[-i \int_0^t V(X_\omega(\tau))\, d\tau\right] : X_\omega(t) = x \right\}$$

It follows that (2.2) is a discrete analog of the Feynman path integral.

Let e_1, e_2, e_3 be the standard unit vectors in $\mathbb{R}^3$ and let $e_{j+3} = -e_j$, $j = 1, 2, 3$, and $e_0 = 0$. A point $x \in Q^3$ then has the six nearest neighbors $x + (a\varepsilon)^{1/2} e_j$, $j = 1, \ldots, 6$, and $x = x + (a\varepsilon)^{1/2} e_0$.

Theorem 2.1. The evolution $U(t, x) = (U_{V,t}g)(x)$ satisfies the initial condition $U(0, x) = g(x)$ and the difference equation

$$U(t + \varepsilon, x) = \sum_{r=0}^{6} a_r [1 - i\varepsilon V(x - (a\varepsilon)^{1/2} e_r)]\, U(t, x - (a\varepsilon)^{1/2} e_r)$$

Proof. For the initial condition we have from (2.2) that

$$U(0, x) = \sum_{\omega} f(\omega) g(x) = g(x)$$

To derive the difference equation, applying (2.2) gives

$$\begin{aligned} U(t+\varepsilon, x) &= \sum \{K_{V,t+\varepsilon}(y, \omega) g(y) : y \in Q^3, y + X_\omega(t+\varepsilon) = x\} \\ &= \sum_{r=0}^{6} \sum_{y,\omega_r} K_{V,t+\varepsilon}(y, \omega_r) g(y) \end{aligned}$$

where in addition to the condition

$$y + X(t+\varepsilon, \omega_r) = x$$

the ω_r satisfy

$$y + X(t, \omega_r) = x - (a\varepsilon)^{1/2} e_r, \qquad r = 0, \ldots, 6$$

If $t = k\varepsilon$, we have

$$\begin{aligned} K_{V,t+\varepsilon}(y, \omega_0) &= f'(\omega_0) \prod_{j=0}^{k} [1 - i\varepsilon V(y + X(j\varepsilon, \omega_0))] \\ &= f'(\omega_0)[1 - i\varepsilon V(x)] \prod_{j=0}^{k-1} [1 - i\varepsilon V(y + X(j\varepsilon, \omega_0))] \\ &= [1 - i\varepsilon V(x)] K_{V,t}(y, \omega_0) \end{aligned}$$

For a fixed $y \in Q^3$ we have by (2.1) that

$$\begin{aligned} \sum_{\omega_0} K_{V,t+\varepsilon}(y, \omega_0) &= [1 - i\varepsilon V(x)] \sum_{\omega_0} K_{V,t}(y, \omega_0) \\ &= [1 - i\varepsilon V(x)] \alpha_0 \sum_{\omega} \{K_{V,t}(y, \omega) : y + X_\omega(t) = x\} \end{aligned}$$

Hence,

$$\sum_{y,\omega_0} K_{V,t+\varepsilon}(y, \omega_0) g(y) = [1 - i\varepsilon V(x)] \alpha_0 U(t, x)$$

In a similar way, for $r = 1, \ldots, 6$, we have

$$\sum_{y,\omega_r} K_{V,t+\varepsilon}(y, \omega_r) g(y) = [1 - i\varepsilon V(x - (a\varepsilon)^{1/2} e_r)] \alpha_r U(t, x - (a\varepsilon)^{1/2} e_r)$$

and the result follows. ■

We say that the amplitude function f is *symmetric* if $\alpha_j = \alpha_{j+3}$, $j = 1, 2, 3$. In the case of a symmetric f, the difference equation in Theorem 2.1 has a special form.

Corollary 2.2. If f is symmetric, then $U(t, x)$ satisfies the equation

$$\frac{U(t+\varepsilon, x)-U(t, x)}{\varepsilon}$$
$$=a\sum_{r=1}^{3}\alpha_r\frac{U(t, x+(a\varepsilon)^{1/2}e_r)+U(t, x-(a\varepsilon)^{1/2}e_r)-2U(t, x)}{a\varepsilon}$$
$$-i\sum_{r=0}^{6}\alpha_r V(x-(a\varepsilon)^{1/2}e_r)U(t, x-(a\varepsilon)^{1/2}e_r) \tag{2.3}$$

Proof. Applying Theorem 2.1 gives

$$U(t+\varepsilon, x)$$
$$=\sum_{r=0}^{6}\alpha_r U(t, x-(a\varepsilon)^{1/2}e_r)$$
$$-i\varepsilon\sum_{r=0}^{6}\alpha_r V(x-(a\varepsilon)^{1/2}e_r)U(t, x-(a\varepsilon)^{1/2}e_r)$$
$$=\left(1-\sum_{r=1}^{6}\alpha_r\right)U(t, x)+\sum_{r=1}^{6}\alpha_r U(t, x-(a\varepsilon)^{1/2}e_r)$$
$$-i\varepsilon\sum_{r=0}^{6}\alpha_r V(x-(a\varepsilon)^{1/2}e_r)U(t, x-(a\varepsilon)^{1/2}e_r)$$
$$=U(t, x)+\sum_{r=1}^{6}\alpha_r[U(t, x-(a\varepsilon)^{1/2}e_r)-U(t, x)]$$
$$-i\varepsilon\sum_{r=0}^{6}\alpha_r V(x-(a\varepsilon)^{1/2}e_r)U(t, x-(a\varepsilon)^{1/2}e_r)$$
$$=U(t, x)+\sum_{r=1}^{3}\alpha_r[U(t, x+(a\varepsilon)^{1/2}e_r)+U(t, x-(a\varepsilon)^{1/2}e_r)-2U(t, x)]$$
$$-i\varepsilon\sum_{r=0}^{6}\alpha_r V(x-(a\varepsilon)^{1/2}e_r)U(t, x-(a\varepsilon)^{1/2}e_r)$$

and the result follows. ■

That (2.3) is a discrete version of a generalized Schrödinger equation may be seen as follows. Suppose V is continuous and U has a twice differentiable extension $u: \mathbb{R}\times\mathbb{R}^3\to\mathbb{C}$. The left side of (2.3) is the forward difference and corresponds to $\partial u/\partial t$. The terms in the first summation on the right side of (2.3) are central differences and correspond to $\partial^2 u/(\partial x^r)^2$. Since V is continuous, letting $\varepsilon\to 0$, each term in the second summation on the right side of (2.3) reduces to $V(x)U(t, x)$. Hence, (2.3) corresponds to the partial differential equation

$$i\frac{\partial u}{\partial t}=ia\sum_{r=1}^{3}\alpha_r\frac{\partial^2 u}{(\partial x^r)^2}+Vu \tag{2.4}$$

If $\alpha_r=i$, $r=1, 2, 3$, $\alpha_0=1-6i$, then (2.4) reduces to Schrödinger's equation. There are many choices for the α_r which result in Schrödinger's equation. However, they must have the form $\alpha_r=ib$, $\alpha_0=1-6ib$, $r=1, 2, 3$, $b>0$. Also note that if $\alpha_r=1/6$, $r=1, 2, 3$, then we obtain a classical probability space and if we replace V by $-iV$, we get the heat equation.

So far we have discussed the three-dimensional quantum random walk $X: T\times\Omega\to Q^3$. It is also instructive to consider the corresponding one-dimensional quantum random walk $X_1: T\times\Omega\to Q$. In the next section, we shall show that X^r, $r=1, 2, 3$, are stochastically dependent isomorphic copies of X_1. In the one-dimensional case, the sample space is given by $\Omega_1=\{-1, 0, 1\}^T$. We then define $X_1: T\times\Omega_1\to Q$ by

$$X_1(k\varepsilon, \omega)=(a\varepsilon)^{1/2}\sum_{j=0}^{k-1}\omega(j\varepsilon)$$

Let α_0, α_1, $\alpha_2\in\mathbb{C}$ and define $f'\colon\{-1, 0, 1\}\to\mathbb{C}$ by $f'(1)=\alpha_1$, $f'(-1)=\alpha_2$, $f'(0)=\alpha_0=1-\alpha_1-\alpha_2$. We then define the amplitude function $f\colon\Omega\to\mathbb{C}$ as before. We also define $K_{V,t}\colon Q\times\Omega\to\mathbb{C}$ and $U_{V,t}g\colon Q\times\Omega\to\mathbb{C}$ in an analogous fashion. If $\alpha_1=\alpha_2$, the difference equation (2.3) now becomes

$$\begin{aligned}&\frac{U(t+\varepsilon, x)-U(t, x)}{\varepsilon}\\&=a\alpha_1\frac{U(t, x+(a\varepsilon)^{1/2})+U(t, x-(a\varepsilon)^{1/2})-2U(t, x)}{a\varepsilon}\\&-i\alpha[V(x+(a\varepsilon)^{1/2})U(t, x+(a\varepsilon)^{1/2})+V(x-(a\varepsilon)^{1/2})U(t, x-(a\varepsilon)^{1/2})]\\&-i\alpha_0V(x)U(x)\end{aligned}$$

If $\alpha_1=i/2$ (or, more generally, $\alpha_1=ib$, $b>0$), this corresponds to the one-dimensional Schrödinger equation.

3. PROPERTIES

If X and f are defined as in Section 2, we call (X, f) a *Schrödinger random walk*. In this section we shall not introduce a potential V, so we are really considering the random motion of a free particle. We shall show that (X, f) has many of the properties of a classical random walk except that the usual probability measure is replaced by the amplitude measure

$f(A)=\sum_{\omega\in A} f(\omega)$. Since a classical random walk is a discrete version of Brownian motion, this shows that the Schrödinger random walk may be regarded as a discrete quantum Brownian motion.

For integers $0\leq s<t\leq N^2$, the *increment* $\Delta_s^t X:\Omega\rightarrow Q^3$ is defined as $\Delta_s^t X=X_{t\varepsilon}-X_{s\varepsilon}$. An increment has three components denoted by

$$(\Delta_s^t X)^r=X_{t\varepsilon}^r-X_{s\varepsilon}^r, \qquad r=1,2,3$$

Notice that

$$(\Delta_s^t X)^r(\omega)=(a\varepsilon)^{1/2}\sum_{j=s}^{t-1}\{\omega^2(j\varepsilon): \omega(j\varepsilon)\neq 0, \omega^1(j\varepsilon)=r\}$$

For $A\subseteq Q^3$, we use the notation

$$f(\Delta_s^t X\in A)=f(\{\omega\in\Omega: \Delta_s^t X(\omega)\in A\})=f[(\Delta_s^t X)^{-1}(A)]$$

It is clear that (X,f) is *stationary*. That is,

$$f(\Delta_s^t X\in A)=f(\Delta_0^{t-s}X\in A)=f(X_{(t-s)\varepsilon}\in A)$$

We next show that increments are *amplitude independent*. That is, if $s_1<t_1\leq s_2<t_2\leq\cdots\leq s_n<t_n$ are integers and $A_1,\ldots,A_n\subseteq Q^3$, then

$$f(\Delta_{s_1}^{t_1}X\in A_1,\ldots,\Delta_{s_n}^{t_n}X\in A_n)=\prod_{j=1}^n f(\Delta_{s_j}^{t_j}X\in A_j)$$

Theorem 3.1. The increments of X are amplitude independent.

Proof. Let $s_1<t_1\leq s_2<t_2$ and let k_j^r, $j=1,2$, $r=1,2,3$, be integers satisfying

$$-(t_j-s_j)\leq k_j^r\leq t_j-s_j$$

We then have

$$\begin{aligned}&f(\Delta_{s_1}^{t_1}X=(a\varepsilon)^{1/2}(k_1^1,k_1^2,k_1^3), \Delta_{s_2}^{t_2}X=(a\varepsilon)^{1/2}(k_2^1,k_2^2,k_3^2))\\&\quad=\sum\{f(\omega): \omega\in A_1^r, \omega\in A_2^r, r=1,2,3\}\end{aligned}$$

where

$$A_i^r=\left\{\omega\in\Omega: \sum_{j=s_i}^{t_i-1}\{\omega^2(j\varepsilon): \omega^1(j\varepsilon)=r\}=k_i^r\right\}$$

It follows from (2.1) that

$$\begin{aligned}&\sum\{f(\omega): \omega\in A_1^r, \omega\in A_2^r, r=1,2,3\}\\&\quad=\sum\{f(\omega): \omega\in A_1^r, r=1,2,3\}\sum\{f(\omega): \omega\in A_2^r, r=1,2,3\}\\&\quad=f(\Delta_{s_1}^{t_1}X=(a\varepsilon)^{1/2}(k_1^1,k_1^2,k_1^3))f(\Delta_{s_2}^{t_2}X=(a\varepsilon)^{1/2}(k_2^1,k_2^2,k_2^3))\end{aligned}$$

We have thus shown that for every $a, b \in Q^3$ we have

$$f(\Delta_{s_1}^{t_1}X=a,\ \Delta_{s_2}^{t_2}X=b)=f(\Delta_{s_1}^{t_1}X=a)f(\Delta_{s_2}^{t_2}X=b) \tag{3.1}$$

Now let $A, B \subseteq Q^3$ with $A=\{a_1, \ldots, a_p\}$, $B=\{b_1, \ldots, b_q\}$. Applying (3.1) and the additivity of the measure f gives

$$\begin{aligned}
f(\Delta_{s_1}^{t_1}X \in A,\ \Delta_{s_2}^{t_2}X \in B) &= f\left(\bigcup_j (\Delta_{s_1}^{t_1}X=a_j) \cap \bigcup_k (\Delta_{s_2}^{t_2}X=b_k)\right)\\
&= f\left(\bigcup_{j,k} (\Delta_{s_1}^{t_1}X=a_j,\ \Delta_{s_2}^{t_2}X=b_k)\right)\\
&= \sum_{j,k} f(\Delta_{s_1}^{t_1}X=a_j,\ \Delta_{s_2}^{t_2}X=b_k)\\
&= \sum_{j,k} f(\Delta_{s_1}^{t_1}X=a_j) f(\Delta_{s_2}^{t_2}X=b_k)\\
&= \sum_j f(\Delta_{s_1}^{t_1}X=a_j) \sum_k f(\Delta_{s_2}^{t_2}X=b_k)\\
&= f(\Delta_{s_1}^{t_1}X \in A) f(\Delta_{s_2}^{t_2}X \in B)
\end{aligned}$$

We have thus proved the result for two increments. The proof for n increments is similar. ■

We now find the amplitude distribution $f_t(x)$ of X, where $t \in T$, $x \in Q^3$, and

$$f_t(x)=f(X_t=x)=\sum_\omega \{f(\omega) : X_t(\omega)=x\}$$

For simplicity, we assume in the remainder of this section that f is symmetric and $\alpha_r=i/2$, $r=1, 2, 3$, $\alpha_0=1-3i$. We have seen in Section 2 that such an amplitude function results in a discrete version of Schrödinger's equation. We then have

$$f(\omega)=\left(\frac{i}{2}\right)^\alpha (1-3i)^\beta \left(\frac{i}{2}\right)^\gamma$$

where

$$\begin{aligned}
\alpha &= |\{j\colon \omega(j\varepsilon)\neq 0,\ \omega^2(j\varepsilon)=1\}|\\
\beta &= |\{j\colon \omega(j\varepsilon)=0\}|\\
\gamma &= |\{j\colon \omega(j\varepsilon)\neq 0,\ \omega^2(j\varepsilon)=-1\}|
\end{aligned}$$

Let k, β be nonnegative integers and $\alpha=(\alpha_1, \alpha_2, \alpha_3)$, $\gamma=(\gamma_1, \gamma_2, \gamma_3)$ be nonnegative integer triples. If

$$\alpha+\beta+\gamma\equiv\alpha_1+\alpha_2+\alpha_3+\beta+\gamma_1+\gamma_2+\gamma_3=k$$

we use the following notation for the multinomial coefficient

$$\binom{k}{\alpha, \beta, \gamma}\equiv\binom{k}{\alpha_1, \alpha_2, \alpha_3, \beta, \gamma_1, \gamma_2, \gamma_3}=\frac{k!}{\alpha_1!\,\alpha_2!\,\alpha_3!\,\beta!\,\gamma_1!\,\gamma_2!\,\gamma_3!}$$

We also use the notation

$$\left(\frac{i}{2}\right)^{\alpha}(1-3i)^{\beta}\left(\frac{i}{2}\right)^{\gamma}=\left(\frac{i}{2}\right)^{\alpha_1+\alpha_2+\alpha_3}(1-3i)^{\beta}\left(\frac{i}{2}\right)^{\gamma_1+\gamma_2+\gamma_3}$$

It follows from the multinomial formula that

$$\sum_{\alpha+\beta+\gamma=k}\binom{k}{\alpha, \beta, \gamma}\left(\frac{i}{2}\right)^{\alpha}(1-3i)^{\beta}\left(\frac{i}{2}\right)^{\gamma}=1 \tag{3.2}$$

If $j=(j_1, j_2, j_3)\in\mathbb{Z}^3$ and $t=k\varepsilon$, then applying (3.2) gives

$$\begin{aligned} f_{k\varepsilon}((a\varepsilon)^{1/2}j)&=\sum_{\omega}\{f(\omega): X_{k\varepsilon}(\omega)=(a\varepsilon)^{1/2}j\} \\ &=\sum_{\substack{\alpha+\beta+\gamma=k\\ \alpha-\gamma=j}}\binom{k}{\alpha, \beta, \gamma}\left(\frac{i}{2}\right)^{\alpha}(1-3i)^{\beta}\left(\frac{i}{2}\right)^{\gamma} \end{aligned} \tag{3.3}$$

In a similar way, the amplitude distribution $\hat{f}_t(x)$ for the one-dimensional Schrödinger random walk X_1 is given by

$$\hat{f}_{k\varepsilon}((a\varepsilon)^{1/2}j_1)=\sum_{\substack{\alpha_1+\beta+\gamma_1=k\\ \alpha_1-\gamma_1=j_1}}\binom{k}{\alpha_1, \beta, \gamma_1}\left(\frac{i}{2}\right)^{\alpha_1}(1-i)^{\beta}\left(\frac{i}{2}\right)^{\gamma_1} \tag{3.4}$$

We now show that the components X^r, $r=1, 2, 3$, of X are isomorphic copies of X_1. Let $f_t^1(x)$ be the amplitude distribution of X^1. That is,

$$f_t^1(x)=\sum\{f(\omega): X_t^1(\omega)=x\}=f(X_t^1=x)$$

As in (3.3), we have

$$f_{k\varepsilon}^1((a\varepsilon)^{1/2}j_1)\sum_{\substack{\alpha+\beta+\gamma=k\\ \alpha_1-\gamma_1=j_1}}\binom{k}{\alpha, \beta, \gamma}\left(\frac{i}{2}\right)^{\alpha}(1-3i)^{\beta}\left(\frac{i}{2}\right)^{\gamma}=S_1S_2$$

where

$$S_1 = \sum_{\substack{\alpha_1+\gamma_1 \le k \\ \alpha_1-\gamma_1=j_1}} \frac{k!}{[k-(\alpha_1+\gamma_1)]!\,\alpha_1!\,\gamma_1!} \left(\frac{i}{2}\right)^{\alpha_1} \left(\frac{i}{2}\right)^{\gamma_1}$$

and

$$S_2 = \sum \binom{k-(\alpha_1+\gamma_1)}{\alpha_2,\ \alpha_3,\ \beta,\ \gamma_2,\ \gamma_3} \left(\frac{i}{2}\right)^{\alpha_2} \left(\frac{i}{2}\right)^{\alpha_3} (1-3i)^{\beta} \left(\frac{i}{2}\right)^{\gamma_2} \left(\frac{i}{2}\right)^{\gamma_3}$$

where the summation in S_2 is over $\alpha_2,\ \alpha_3,\ \beta,\ \gamma_2,\ \gamma_3$ satisfying

$$\alpha_2+\alpha_3+\beta+\gamma_2+\gamma_3 = k-(\alpha_1+\gamma_1)$$

It follows from the multinomial formula that

$$S_2 = (1-i)^{k-(\alpha_1+\gamma_1)}$$

Hence, letting $\beta' = k-(\alpha_1+\gamma_1)$ gives

$$f^1_{k\varepsilon}((a\varepsilon)^{1/2} j_1) = \sum_{\substack{\alpha_1+\beta'+\gamma_1=k \\ \alpha_1-\gamma_1=j_1}} \binom{k}{\alpha_1,\ \beta',\ \gamma_1} \left(\frac{i}{2}\right)^{\alpha_1} (1-i)^{\beta'} \left(\frac{i}{2}\right)^{\gamma_1}$$

This is the same as (3.4), so X^1 has the same amplitude distribution as X_1. The same result holds for X^2 and X^3.

We have shown that the components X^r of X are one-dimensional Schrödinger random walks. We now demonstrate the surprising result that the X^r, $r=1, 2, 3$, are not amplitude independent. Indeed,

$$f_\varepsilon[(a\varepsilon)^{1/2}(1, 1, 1)] = f(X^1_\varepsilon = (a\varepsilon)^{1/2}, X^2_\varepsilon = (a\varepsilon)^{1/2}, X^3_\varepsilon = (a\varepsilon)^{1/2}) = 0$$

since otherwise we would have an $\omega \in \Omega$ satisfying

$$\omega(0) = (1, 1) = (2, 1) = (3, 1)$$

which is impossible. On the other hand,

$$\begin{aligned} &f^1_\varepsilon((a\varepsilon)^{1/2}) f^2_\varepsilon((a\varepsilon)^{1/2}) f^3_\varepsilon((a\varepsilon)^{1/2}) \\ &\quad = f(X^1_\varepsilon = (a\varepsilon)^{1/2}) f(X^2_\varepsilon = (a\varepsilon)^{1/2}) f(X^3_\varepsilon = (a\varepsilon)^{1/2}) = \left(\frac{i}{2}\right)^3 = -\frac{i}{8} \neq 0 \end{aligned}$$

Thus, in general, we have

$$f_t(x) \neq f^1_t(x^1) f^2_t(x^2) f^3_t(x^3)$$

Because of this, we cannot find the three-dimensional Schrödinger distribution using one-dimensional Schrödinger distributions.

If $Y:\Omega\to\mathbb{C}$, we define the *amplitude expectation* of Y by

$$E(Y)=\sum_{\omega} Y(\omega)f(\omega)$$

If Y has the values $y_1,\ldots,y_n$, it follows that

$$E(Y)=\sum y_j f(Y=y_j) \tag{3.5}$$

Let $F, G: Q^3\to\mathbb{C}$ and let $0\le s_1<t_1\le s_2<t_2\le N^2$ be integers. Since $\Delta_{s_1}^{t_1}X$ and $\Delta_{s_2}^{t_2}X$ are amplitude independent, it follows as in classical probability theory that

$$E[F(\Delta_{s_1}^{t_1}X)G(\Delta_{s_2}^{t_2}X)]=E[F(\Delta_{s_1}^{t_1}X)]E[G(\Delta_{s_2}^{t_2}X)]$$

For $t\in T$, we define the *amplitude characteristic function* $\Phi_t:\mathbb{R}^3\to\mathbb{C}$ by

$$\Phi_t(y)=E(e^{iy\cdot X_t})=E[\exp i(y_1X_t^1+y_2X_t^2+y_3X_t^3)]$$

The next result derives an explicit expression for $\Phi_t(y)$ and uses this expression to find another form for $f_t(x)$.

Theorem 3.2. (a) The amplitude characteristic function is given by

$$\Phi_{k\varepsilon}(y)=\left[(1-3i)+i\sum_{r=1}^{3}\cos y_r(a\varepsilon)^{1/2}\right]^k$$

(b) The amplitude distribution has the form

$$f_{k\varepsilon}((a\varepsilon)^{1/2}j)=\left(\frac{(a\varepsilon)^{1/2}}{2\pi}\right)^3\iiint\Phi_{k\varepsilon}(y)\exp[-i(a\varepsilon)^{1/2}y\cdot j]\,d^3y$$

where the integrals have limits $-\pi/(a\varepsilon)^{1/2}$ and $\pi/(a\varepsilon)^{1/2}$.

Proof. (a) Applying independence and stationarity of increments, we have

$$\begin{aligned}
\Phi_{k\varepsilon}(y)&=E[\exp(iy\cdot X_{k\varepsilon})]\\
&=E\{\exp iy\cdot[(X_\varepsilon-X_0)+(X_{2\varepsilon}-X_\varepsilon)+\cdots+(X_{k\varepsilon}-X_{(k-1)\varepsilon})]\}\\
&=E(\exp iy\cdot X_\varepsilon\exp iy\cdot\Delta_\varepsilon^{2\varepsilon}X\cdots\exp iy\cdot\Delta_{(k-1)\varepsilon}^{k\varepsilon}X)\\
&=E(\exp iy\cdot X_\varepsilon)E(\exp iy\cdot\Delta_\varepsilon^{2\varepsilon}X)\cdots E(\exp iy\cdot\Delta_{(k-1)\varepsilon}^{k\varepsilon}X)\\
&=[E(\exp iy\cdot X_\varepsilon)]^k
\end{aligned}$$

Now

$$\begin{aligned} E(\exp iy\cdot X_\varepsilon) &= \sum_\omega [\exp iy\cdot X_\varepsilon(\omega)]f(\omega) \\ &= (1-3i)+\frac{i}{2}\sum_{r=1}^{3}\exp[iy_r(a\varepsilon)^{1/2}]+\frac{i}{2}\sum_{r=1}^{3}\exp[-iy_r(a\varepsilon)^{1/2}] \\ &= (1-3i)+i\sum_{r=1}^{3}\cos y_r(a\varepsilon)^{1/2} \end{aligned}$$

and the result follows.

(b) Applying (3.5), we have for $t=k\varepsilon$

$$\Phi_t(y)=\sum_{r\in\mathbb{Z}^3}\{\exp[i(a\varepsilon)^{1/2}y\cdot r]\}f_t((a\varepsilon)^{1/2}r)$$

which is a finite Fourier series. Taking the inverse transform gives the result. ■

It is instructive to see what happens to the results of Theorem 3.2 when $\varepsilon\to 0$. In this case we have

$$\Phi_t(y)\approx\left(1-\frac{ia\varepsilon\|y\|^2}{2}\right)^k\approx(e^{-ia\varepsilon\|y\|^2/2})^k=e^{-iat\|y\|^2/2}$$

Except for the factor i, this is the same as the characteristic function for Brownian motion. Moreover, for $k\neq 0$, we have

$$\begin{aligned} &f_{k\varepsilon}((a\varepsilon)^{1/2}j) \\ &\approx\left(\frac{(a\varepsilon)^{1/2}}{2\pi}\right)^3\iiint\exp(-\tfrac{1}{2}iak\varepsilon\|y\|^2)\exp[-i(a\varepsilon)^{1/2}y\cdot j]\,d^3y \\ &=\left(\frac{(a\varepsilon)^{1/2}}{2\pi}\right)^3\left(\exp\frac{i\|j\|^2}{2k}\right) \\ &\quad\times\iiint\exp\left[-i\left\|\left(\frac{a\varepsilon k}{2}\right)^{1/2}y+\frac{j}{(2k)^{1/2}}\right\|^2\right]d^3y \end{aligned}$$

where the integrals have limits $-\pi/(a\varepsilon)^{1/2}$ and $\pi/(a\varepsilon)^{1/2}$. Letting

$$z=\left(\frac{a\varepsilon k}{2}\right)^{1/2}y+\frac{j}{(2k)^{1/2}}$$

gives

$$f_{k\varepsilon}((a\varepsilon)^{1/2}j)=\frac{1}{\pi^3(2k)^{3/2}}e^{i\|j\|^2/2k}\iiint e^{-i\|z\|^2}\,d^3z$$

where the integrals have limits $-\pi(k/2)^{1/2}+j_r/(2k)^{1/2}$ and $\pi(k/2)^{1/2}+j_r/(2k)^{1/2}$, $r=1, 2, 3$.

4. DIRAC RANDOM WALKS

Let C_4 be the complex Clifford algebra with identity 1 and generators γ^μ, $\mu=0, 1, 2, 3$, satisfying $(\gamma^0)^2=1$, $(\gamma^j)^2=-1$, $j=1, 2, 3$, and $\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=1$, $\mu\neq\nu$. Denoting the 2×2 identity matrix by I, we can represent C_4 as a Clifford algebra of 4×4 matrices in which

$$1=\begin{bmatrix} I & 0\\ 0 & I\end{bmatrix},\qquad \gamma^0=\begin{bmatrix} I & 0\\ 0 & -I\end{bmatrix},\qquad \gamma^j=\begin{bmatrix} 0 & \sigma^j\\ -\sigma^j & 0\end{bmatrix},\qquad j=1, 2, 3$$

where σ^j, $j=1, 2, 3$, are the Pauli matrices

$$\sigma^1=\begin{bmatrix} 0 & 1\\ 1 & 0\end{bmatrix},\qquad \sigma^2=\begin{bmatrix} 0 & -i\\ i & 0\end{bmatrix},\qquad \sigma^3=\begin{bmatrix} 1 & 0\\ 0 & -1\end{bmatrix}$$

However, our work will not depend on a particular representation. If a is a four-vector, we use the notation $\not{a}=\gamma^\mu a_\mu$, where the summation convention is employed. If

$$\partial=\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^2}\right)$$

then Dirac's equation has the form

$$(i\not{\partial}-\not{A}(t, x)-m1)\psi(t, x)=0$$

where $A(t, x)$ is the charge times the four-potential, m is the mass, and $\psi:\mathbb{R}^4\to C_4$.

As in Section 2, we let $\varepsilon=1/N$ and define the discrete timeline

$$T=\{0, \varepsilon, 2\varepsilon, \ldots, N^2\varepsilon\}$$

The discrete space lattice is now defined as $Q^3=\varepsilon\mathbb{Z}^3$. The sample space $\Omega=S^T$ is the same as in Section 2. For $r=1, 2, 3$, we now define $X^r: T\times\Omega\to Q^3$ by

$$X^r(k\varepsilon, \omega)=\varepsilon\sum_{j=0}^{k-1}\{\omega^2(j\varepsilon): \omega(j\varepsilon)\neq 0, \omega^1(j\varepsilon)=r\}$$

if $k \neq 0$ and $X^r(0, \omega) = 0$. Finally, we define the random walk $X: T \times \Omega \to Q^3$ by

$$X(t, \omega) = (X^1(t, \omega), X^2(t, \omega), X^3(t, \omega))$$

Besides the change in the definition of Q^3 and X, the main difference between the Dirac and Schrödinger random walks is that we now define the amplitude f as a C_4-valued function. Let $\alpha_0, \ldots, \alpha_6 \in C_4$ satisfy $\sum_{j=0}^{6} \alpha_j = 1$ and define $f': S \to C_4$ by $f'(0) = \alpha_0$, $f'((j, 1)) = \alpha_j$, $f'((j, -1)) = \alpha_{j+3}$, $j = 1, 2, 3$. As in Section 2, we define the *amplitude function* $f: \Omega \to C_4$ by

$$f(\omega) = f'[\omega(N^2\varepsilon)] f'[\omega((N^2-1)\varepsilon)] \cdots f'[\omega(\varepsilon)] f'[\omega(0)]$$

It again follows by induction that for any distinct $r_1\varepsilon, \ldots, r_n\varepsilon \in T$ we have

$$\sum_{\omega} f'[\omega(r_1\varepsilon)] \cdots f'[\omega(r_n\varepsilon)] = 1 \tag{4.1}$$

Corresponding to the four-potential $A(t, x)$ and $t \in T$, $y \in Q^3$, $\omega \in \Omega$, we define the C_4 element

$$V(t, y + X_\omega(t)) = 1 - i\varepsilon\gamma^0[\not{A}(t, y + X_\omega(t)) + m1]$$

For $t \in T$, define the *propagator* $K_{V,t}: Q^3 \times \Omega \to C_4$ as follows:

$$\begin{aligned} K_{V,t}(y, \omega) = {} & f'[\omega(N^2\varepsilon)] \cdots f'[\omega(t-\varepsilon)] V[t-\varepsilon, y + X_\omega(t-\varepsilon)] \\ & \times f'[\omega(t-2\varepsilon)] V[t-2\varepsilon, y + X_\omega(t-2\varepsilon)] \cdots f'[\omega(0)] V(0, y) \end{aligned}$$

for $k \neq 0$ and $K_{V,0}(y, \omega) = f(\omega)$. Let $g: \mathbb{R}^3 \to C_4$ be a function which we regard as the initial amplitude. The *evolution* $U_{V,t}g: Q^3 \to C_4$ is defined by

$$(U_{V,t}g)(x) = \sum \{K_{V,t}(y, \omega) g(y): y \in Q^3, y + X_t(\omega) = x\}$$

As in Section 2, we define the vectors $e_j \in \mathbb{R}$, $j = 0, \ldots, 6$.

Theorem 4.1. The evolution $\Psi(t, x) = (U_{V,t}g)(x)$ satisfies the initial condition $\Psi(0, x) = g(x)$ and the difference equation

$$\Psi(t+\varepsilon, x) = \sum_{r=0}^{6} \alpha_r V(t, x - \varepsilon e_r) \Psi(t, x - \varepsilon e_r)$$

Proof. For the initial condition we have from (4.1) that

$$\Psi(0, x) = \sum_{\omega} f(\omega) g(x) = g(x)$$

To derive the difference equation, we have

$$\Psi(t+\varepsilon, x) = \sum_{r=0}^{6} \sum_{y,\omega_r} K_{V,t+\varepsilon}(y, \omega_r) g(y)$$

where the ω_r satisfy the conditions stated in the proof of Theorem 2.1. If $t+\varepsilon \in T$, we have

$$\begin{aligned} &K_{V,t+\varepsilon}(y, \omega_0) \\ &\quad = f'[\omega_0(N^2\varepsilon)] \cdots f'[\omega_0(t)] V(t, x) f'[\omega_0(t-\varepsilon)] \\ &\qquad \times V[t-\varepsilon, y + X_{\omega_0}(t-\varepsilon)] \cdots f'(\omega_0) V(0, y) \end{aligned}$$

For a fixed $y \in Q^3$, we have by (4.1) that

$$\begin{aligned} &\sum_{\omega_0} K_{V,t+\varepsilon}(y, \omega_0) \\ &\quad = \alpha_0 V(t, x) \sum_{\omega_0} f'[\omega_0(t-\varepsilon)] V[t-\varepsilon, y + X_{\omega_0}(t-\varepsilon)] \cdots f'(\omega_0) V(0, y) \\ &\quad = \alpha_0 V(t, x) \sum_{\omega} \{K_{V,t}(y, \omega) : y + X_\omega(t) = x\} \end{aligned}$$

Hence,

$$\sum_{y,\omega_0} K_{V,t+\varepsilon}(y, \omega_0) g(y) = \alpha_0 V(t, x) \Psi(t, x)$$

In a similar way, for $r = 1, \ldots, 6$, we have

$$\sum_{y,\omega_r} K_{V,t+\varepsilon}(y, \omega_r) g(y) = \alpha_r V(t, x - \varepsilon e_r) \Psi(t, x - \varepsilon e_r)$$

and the result follows. ∎

Corollary 4.2. $\Psi(t, x)$ satisfies the equation

$$\begin{aligned} &\frac{\Psi(t+\varepsilon, x) - \Psi(t, y)}{\varepsilon} \\ &\quad = \sum_{r=1}^{3} \alpha_r \left[\frac{\Psi(t, x - \varepsilon e_r) - \Psi(t, x)}{\varepsilon}\right] + \sum_{r=1}^{3} \alpha_{r+3} \left[\frac{\Psi(t, x + \varepsilon e_r) - \Psi(t, x)}{\varepsilon}\right] \\ &\qquad - i \sum_{r=0}^{6} \alpha_r \gamma^0 [\not{A}(t, x - \varepsilon e_r) + m1] \Psi(t, x - \varepsilon e_r) \end{aligned} \tag{4.2}$$

Proof. Applying Theorem 4.1 gives

$$\begin{aligned}
&\Psi(t+\varepsilon, x)\\
&\quad=\alpha_0[1-i\varepsilon\gamma^0(\mathcal{A}(t,x)+m1)]\Psi(t,x)\\
&\qquad+\sum_{r=1}^{6}\alpha_r[1-i\varepsilon\gamma^0(\mathcal{A}(t,x-\varepsilon e_r)+m1)]\Psi(t,x-\varepsilon e_r)\\
&\quad=\left(1-\sum_{r=1}^{6}\alpha_r\right)\Psi(t,x)+\sum_{r=1}^{6}\alpha_r\Psi(t,x-\varepsilon e_r)\\
&\qquad-i\varepsilon\sum_{r=0}^{6}\alpha_r\gamma^0[\mathcal{A}(t,x-\varepsilon e_r)+m1]\psi(t,x-\varepsilon e_r)
\end{aligned}$$

and the result follows. ■

That (4.2) is a discrete version of a generalized Dirac equation may be seen as follows. Suppose $A(t, x)$ is continuous and Ψ has a differentiable extension $\psi: \mathbb{R}^4 \to C_4$. Letting $\varepsilon \to 0$, we conclude that (4.2) corresponds to the partial differential equation

$$\frac{\partial\psi}{\partial t}=\sum_{r=1}^{3}(\alpha_{r+3}-\alpha_r)\frac{\partial\psi}{\partial x^r}-i\gamma^0[\mathcal{A}(t,x)+m1]\psi \tag{4.3}$$

Multiplying both sides of (4.3) by $i\gamma^0$ gives

$$\left[i\left(\gamma^0\frac{\partial}{\partial t}+\sum_{r=1}^{3}\gamma^0(\alpha_r-\alpha_{r+3})\frac{\partial}{\partial x^r}\right)-\mathcal{A}(t,x)-m1\right]\psi(t,x)=0 \tag{4.4}$$

Now (4.4) reduces to Dirac's equation if and only if

$$\gamma^0(\alpha_r-\alpha_{r+3})=\gamma^r, \qquad r=1,2,3 \tag{4.5}$$

There are many ways to realize (4.5). Three of the simplest are the following:

$$\begin{array}{llll}
\alpha_r=\gamma^0\gamma^r, & r=1,2,3, & \alpha_r=0, & r=4,5,6\\
\alpha_r=-\gamma^0\gamma^r, & r=4,5,6, & \alpha_r=0, & r=1,2,3\\
\alpha_r=\frac{1}{2}\gamma^0\gamma^r, & r=1,2,3, & \alpha_r=-\frac{1}{2}\gamma^0\gamma^r, & r=4,5,6
\end{array}$$

A realization of (4.5) that becomes especially simple in the matrix representation is given by

$$\begin{aligned} \alpha_r &= \tfrac{1}{2}(\gamma^0 - 1)\gamma^r, \qquad r = 1, 2, 3 \\ \alpha_r &= \tfrac{1}{2}\gamma^r(\gamma^0 - 1), \qquad r = 4, 5, 6 \end{aligned} \tag{4.6}$$

In the matrix representation, (4.6) becomes

$$\alpha_r = \begin{bmatrix} 0 & 0 \\ \sigma^r & 0 \end{bmatrix}, \qquad r = 1, 2, 3$$

$$\alpha_r = \begin{bmatrix} 0 & -\sigma^r \\ 0 & 0 \end{bmatrix}, \qquad r = 4, 5, 6$$

If X and f are given as in this section, we call (X, f) a *Dirac random walk*. As in Section 3, this corresponds to a free (and in this case) massless particle. We now briefly summarize the properties of (X, f). Using arguments similar to those in Section 3, it can be shown that (X, f) is stationary with amplitude-independent increments. As in Section 3, we define the amplitude expectation E and the amplitude characteristic function $\Psi_t(y)$. In the next theorem we use the following notation. For $y \in \mathbb{R}^3$ we let

$$e^{i\varepsilon y} = (e^{i\varepsilon y^1}, e^{i\varepsilon y^2}, e^{i\varepsilon y^3})$$

We denote $(1, 1, 1) \in \mathbb{R}^3$ by 1 and σ stands for $(\sigma^1, \sigma^2, \sigma^3)$. The proof of the next theorem is similar to that of Theorem 3.2.

Theorem 4.3. Suppose the α_r are given by (4.6). (a) The amplitude characteristic function is given by

$$\Phi_{k\varepsilon}(y) = \begin{bmatrix} I & (1 - e^{-i\varepsilon y}) \cdot \sigma \\ (e^{i\varepsilon y} - 1) \cdot \sigma & I \end{bmatrix}^k$$

(b) The amplitude distribution has the form

$$f_{k\varepsilon}(\varepsilon j) = \left(\frac{\varepsilon}{2\pi}\right)^3 \iiint \Phi_{k\varepsilon}(y)\, e^{-i\varepsilon y \cdot j}\, d^3 y$$

where the integrals have limits $-\pi/\varepsilon$ and π/ε.

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